Sequential Stability of the Constant Cost Dynamic Lot Size Model - Searching for Monotonicity

Revised version

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The set of cost inputs for which an optimal solution of the dynamic lot size model remains valid is called stability region. The size of this region may be viewed as a measure of robustness of a solution. It is an expectation that the stability regions shrink with growing time horizons and that they are monotonous in this sense. In the present paper several sufficient conditions implying monotonicity will be studied. The conditions cover the existence of planning and forecast horizons and generalize the results of a previous paper in which monotonicity results were presented for the case of ordinary planning horizons.

Die Menge der Kostenparameter, für die eine optimale Lösung des dynamischen Losgrößenmodells optimal bleibt wird hier Stabilitätsregion genannt. Die Größe einer solchen Menge kann als Maß der Robustheit einer Lösung angesehen werden. Es ist zu erwarten, daß die Stabilitätsregionen mit wachsendem Zeithorizont schrumpfen und daß sie in diesem Sinne monoton sind. In der vorliegenden Arbeit werden verschiedene hinreichende Bedingungen für diese Monotonie untersucht. Die Bedingungen setzen unter anderem die Existenz von Planungs- und Vorhersage-Horizonten voraus und verallgemeinern so Ergebnisse einer früheren Arbeit, in der Aussagen für gewöhnliche Planungs-Horizonte vorgestellt wurden.

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1. The Stability Problem for the Dynamic Lot Size Model

1.1. The Stability Research for the Dynamic Lot Size Model

The dynamic lot size model [13] has been studied in several papers of the first author and other researchers [5-12] to answer the question for which cost inputs an optimal solutions remains valid. First, stability regions, i. e. sets of cost inputs for which a solution is optimal, were determined by using the dynamic programming solution process [6-10]. The size of the stability region can be regarded as a measure of the robustness of decisions and a high value is economically favorable. An efficient algorithm for the generation of such regions has been discussed in [5]. The regions found in [6-11], however, covered only subsets of the stability region. Later a complete analytic expression for the stability regions has been found in [3,11] by the application of the properties of the minimal inventory cost from [2]. In [11] first attempts have been also made to analyse the behavior of the stability region for growing time horizons t, in other words, for growing planning intervals covering the periods $1,2,...,t$. In this paper the analysis will be continued and several results from [11] will be generalized.

This analysis is motivated by the following observation: Planning and forecast horizons [1] provide an optimal time horizon in dynamic lot size models because the optimal decision for the planning horizon is part of a long run optimal decision. The cost parameters used in the models are certainly approximate and their final values might be different. If the final values belong to the stability region this difference will not affect the optimality of decisions. A large size of the stability region is therefore obviously favorable. The planning horizon as the best time horizon for a model has, however, not necessarily the largest stability region as it will be shown later. With this respect other periods might serve better as time horizons. Hence, the determination of the time horizon can be regarded as a problem with the two criteria "optimality" and "stability", and it is worth checking the behavior of the stability regions.

Since with growing time horizons of a model optimal decisions depend on an increasing number of inputs the stability regions should certainly shrink. Let the notation $S(m,n)$ be used for stability regions of an optimal solution for the periods $m,m+1,...,n$. Then the relations

$$S(m,n) \subset S(r,t)\ ^{1)},\quad S(m,n) \subset S(m,r-1) \cap S(r,n) \quad \text{or} \quad S(m,n) = S(m,r-1) \cap S(r,n)$$

for $1 \leq m \leq r \leq t \leq n$ are expected to hold. The stability region of an optimal solution for the periods $m,m+1,...,n$ is then a subset of the regions for solutions which are optimal for $m,m+1,...,r-1$ and for $r,r+1,...,n$, respectively. In an extreme case it might coincide with the

[1] The inclusion symbol covers in this paper the case of identical sets
intersection of regions. Then the tightest bounds of the regions for these two subproblems determine the bounds of the stability region for the original problem. Such pairs of sets *S(m,n)* and *S(r,t)* are called *monotonous* and the triples *S(m,n), S(m,r-1), S(r,n)* will be called *monotonous* and *strictly monotonous*, respectively. The paper will aim at searching such (strictly) monotonous pairs and triples.

It has been found (comp. [11]) that generally with an increasing time horizon the stability regions might not shrink and monotonous pairs will not exist. The sufficient conditions for the identification of monotonous and strictly monotonous pairs and triples from [11] require the existence of ordinary planning horizons.

In this paper some more efforts are made to analyse this sequential behavior of the stability regions. First, the regions are also shown to be shrinking if so-called planning and forecast horizons exist and the time horizon of the initial problem exceeds the forecast horizon. Secondly, the conditions for the identification of (strictly) monotonous triples are also generalized to the case of planning and forecast horizons. Thirdly, the existence of strictly monotonous triples is proved for the case that a certain stability set of the ordinary planning horizon is sufficiently large. Hence, monotonous and strictly monotonous triples can be found under more general conditions than in [11], new interpretations of the sufficient conditions are presented and some more counterexamples illustrate the conditions used in the assertions.

1.2. The Model and its Solution

The one-product dynamic lot size model covering a planning interval with periods $1,2,...,n$, where n is also called time horizon of the model, consists in determining order variables x_t and inventory variables y_t for one product and all periods $t=1,2,...,n$. When ordering a nonzero amount the setup cost s appears. Storing one item for one period costs h units. Under the assumptions that end period inventory is regarded, the demand $d_t > 0$ of every period is to be satisfied, the stock of that product is zero before and after the planning interval, and planning interval cost is to be minimized, many authors gave the following model:

Nonnegativity of the variables: $x_t \geq 0, y_t \geq 0, t=1,2,...,n.$ (1)

Zero stock before and after the planning interval: $y_o = y_n = 0.$ (2)

Demand satisfaction: $y_t = y_{t-1} + x_t - d_t, t=1,2,...,n.$ (3)

Cost minimization: $\sum_{t=1}^{n}(s \text{ sign } x_t + h\, y_t) \rightarrow min.$ (4)

The model will be denoted by M and if the first and last period are significant by $M(m,n)$ with the first period m and the last period $n \geq m$.
In this paper the techniques used to determine an optimal solution for the model (1) - (4) do not play the major role. It will be assumed that optimal solutions can be found by dynamic programmming procedures based on the recursion of the type

$$f_t = c_{i(t)t} + f_{i(t)-1} = min \{ c_{it} + f_{i-1}: 1 \leq i \leq t\}, f_o = 0, \quad (5)$$

with f_t as minimal cost for the first t periods and c_{it} as sum of the setup cost at period i and of the holding cost for the periods $i,i+1,...,t$. The periods $i(t)$ which cover the last setup for every time horizon t are called regenerations points and the relation $i(t) \leq i(t+1)$ has been proved by many authors. If $i(t)$ is not unique the largest value will be used. Because of this rule always exactly one optimal solution will be generated, although there may exist more than one optimal solution. The recursion (5) and the properties of regeneration points are discusseded in many standard textbooks of Operations Research and Production Planning (comp. [5]). In the last years more efficient algorithms for the dynamic lot size model have been developed (comp. [12]).

The case of $i(r)=r$ is of special interest, since it splits the problem $M(1,n)$ into two independent subproblems $M(1,r-1)$ and $M(r,n)$, if $n \geq r$. The period $r-1$ is called *(ordinary) planning horizon* in the lot size literature, since the next period r will be always a setup period, no matter which time horizon $n \geq r$ is regarded. More generally, a period $r-1$ is called *planning horizon for the forecast horizon t,* if r is setup period in every optimal solution for all models with a time horizon $n \geq t$ (comp. [1]).

Since a feasible solution is uniquely determined by the order values x_t the symbol x will be used to denote feasible or optimal solutions. If the starting and end periods are significant, the notation $x(m,n)$ will denote optimal solutions for $M(m,n)$. A solution is also characterized by the periods in which the goods are ordered. These periods are called *setup periods.*

1.3. Stability Regions

If an optimal solution x is found, it is of interest to know for which parameters s and h it will remain valid. Such a set will be called *stability region*. Since the cost inputs can be normalized by dividing both parameters by the value h only the stability region S for the setup cost input is to be discussed in this and the next sections.

One approach to determine the set S consists in the following idea (comp. [2]): Let solutions with a fixed number of setups k be introduced and let $H(n,k)$ be the minimal inventory cost for the case $h=1$ in a model with n periods and k setups. How such values can be found is at the moment of no interest. The problem of determining an optimal solution is then also provided by

$$sk + H(n,k) \rightarrow min, \quad (6)$$
$$1 \leq k \leq n. \quad (7)$$

The optimal number $k(s)$, which may be not unique for a problem, provides the number of setups of an optimal solution. Set S is characterized in the next theorem.

Theorem 1 ([3, 11]): The (setup cost) stability region of an optimal solution for the dynamic lot size model is provided by the following interval

$$s \in S \Leftrightarrow s^- = H(n,k(s)) - H(n,k(s)+1) \leq s \leq s^+ = H(n,k(s)-1) - H(n,k(s)), \quad (8)$$

where one of the bounds might not exist.

The relation (8) provides a typical marginal property: The setup cost input can vary within the range of the differences in the minimal inventory cost occuring if the number of setups is changed by one.

Example 1: Let $n=3$, $d = (3,2,1)$, $s=2.5$ and $h=1$. The optimal solution for this problem is $x = (3, 3, 0)$ and $k(2.5) = 2$, $H(3,2) = 1$. Then $H(3,1) = 4$, since one setup only implies the demand of the periods 2 and 3 to be ordered at the first period, and $H(3,3) = 0$. It follows then that $S(1,3)$ is provided by

$$s^- = 1 = 1\text{-}0 \leq s \leq 4\text{-}1 = s^+ = 3 \Rightarrow S(1,3) = [1,3].$$

The stability region can also be found explicitly: For the optimal solution the total cost should satisfy

$$2s + 1 \leq 3s, \; 2s + 1 \leq s + 4, \; 2s + 1 \leq 2s + 2 \Rightarrow 1 \leq s \leq 3.$$

As a consequence of the relation (8) R_+ can be covered by a finite number of stability regions which correspond to certain optimal solutions. Tab. 1 illustrates this property for example 1.

Tab. 1 about here

Remark: The model introduced in section 1.2. is given for positive demand. The same stability results can also be obtained for nonnegative demand values. Then, however, either the recursion (5) and the minimal inventory cost $H(n,k)$ have to be redefined, or zero demand values have to be replaced by sufficiently small ε. Details of such a solution approach will not be discussed here.

Let two other examples be considered:

Example 2: Let $n=6$, $d = (3,2,1,2,2,3)$ and $s=2.5$. Then the optimal solution is $x = (3,3,0,4,0,3)$ and $S(1,6) = [2,3]$.

Example 3: Let $n=3$, $d = (2,2,3)$ and $s=2.5$, i. e. the last three periods of the example 2 are regarded. Then the optimal solution is $x = (4,0,3)$ and $S(4,6) = [2,6]$.

It follows from the previous examples that $S(1,3) \cap S(4,6) = [1,3] \cap [2,6] = [2,3] = S(1,6)$ holds. Whether such a relation is true in general will be the point of the next section.

2. Sequential Stability

2.1. More Definitions and Examples

When the time horizon is growing a series of stability regions and appropriate optimal solutions arises. As it was pointed out in section 1.1. a pair of stability regions $S(r,t)$, $S(m,n)$ of optimal solutions for dynamic lot size models $M(r,t)$ and $M(m,n)$, $m \le r \le t \le n$ is called *monotonous* if the subsequent region is a subset of the previous set. A triple of stability regions associated with the periods $m \le r \le n$ is called *monotonous* if

$$S(m,n) \subset S(m,r\text{-}1) \cap S(r,n)$$

holds and *strictly monotonous* if

$$S(m,n) = S(m,r\text{-}1) \cap S(r,n)$$

is fulfilled. If there is a strictly monotonous triple the stability region $S(m,n)$ coincides with the intersection of regions for two subsequent subproblems $M(m,r\text{-}1)$ and $M(r,n)$ and is determined by the tightest bounds of the two regions. A pair of optimal solutions $x(m,n)$, $x(r,t)$ with $m \le r \le t \le n$ is called *monotonous* if the order and inventory values of the solutions coincide for the common periods. For this case the notation $x(r,t) \subset x(m,n)$ will be used. It is obvious that for monotonous pairs of solutions the periods m, r and $t+1$, if $t < n$, are setup periods.

Lemma 1: Let the dynamic lot size model $M(m,n)$ be considered and let the period r-1 be a planning horizon for the forecast horizon t. Then $x(m,r-1) \subset x(m,n)$ and $x(r,n) \subset x(m,n)$ hold for all $n \geq t$.

Proof: According to the definition the period r is a setup period for all models $M(m,n)$. Then the parts of the solution $x(m,n)$ covering the periods $m,m+1,...,r-1$ and $r,r+1,...,n$ must be optimal for the subproblems $M(m,r-1)$ and $M(r,n)$, respectively. □

Two more examples illustrate the various situations occuring for the stability regions and optimal solutions.

Example 4: Let $d = (17,9,12,10,9,7,9,5)$ and $s=20$. Then the stability regions $S(1,t)$ for $t=1,2,...,8$ are as provided in Tab. 2 and in Fig. 1: The horizontal and vertical axes of the figures cover the growing number of periods and the setup cost values s, respectively. The appropriate stability regions are represented by vertical lines. The lower and upper bounds of these lines are also connected to explain which variations of the regions around the initial cost input $s=20$ can be observed. The appropriate optimal solutions are shown in Tab. 2.

Tab. 2 and Fig. 1 about here

Remark: It can be seen that the nonmonotonous changes of the regions are accompanied by solutions which revise previously planned values and are nonmonotonous as well. However, the stability regions and optimal solutions for the periods *6* and *8* form monotonous pairs.

Below the sequential stability regions are analysed under the presence of the planning and forecast horizons. The reason for such assumption is not only mathematically based but also empirically clear: It guarantees also monotonous pairs of optimal solutions; and the monotonicity for stability regions certainly makes only sense for such pairs of solutions.

Example 5: $d = (3,3,...)$ and $s = 10$. It can be seen that even stationary demand will not guarantee the monotonicity of optimal solutions and stability regions (compare Fig. 2 and Tab. 3).

Fig. 2 and Tab. 3 about here

The optimal solutions, the minimal costs divided by the number of periods, and the stability regions are contained in Tab. 3. Fig. 2 shows again the variation of the stability regions around the cost input $s=10$. Additionally, the values of the per-period minimal cost f_t/t are represented by a broken line . The relation between $S(1,t)$ and f_t/t will be discussed in section 2.4.

2.2. Monotonous Pairs and Triples of Stability Regions

Lemma 2: Let $M(m,n)$ be considered, let $r-1$ be a planning horizon for the forecast horizon t, and let $n \geq t$. Then

$$S(m,n) \subset S(m,r-1) \text{ and } S(m,n) \subset S(r,n) \tag{9}$$

hold for all $n \geq t$, i. e. monotonous pairs of stability regions exist.

Proof: Let $s' \in S(m,n)$. The optimal solution $x(m,n)$ is the same as that for the initial parameter s. According to lemma 1 $x(m,r-1)$ and $x(r,n)$ belong to $x(m,n)$ and are therefore optimal for $M(m,r-1)$ and $M(r,n)$ with the setup cost input s', respectively. □

Monotonicity results may also be obtained if no explicit planning and forecast horizons are given. Let T denote the set of setup periods of an optimal solution for the problem $M(1,n)$.

Lemma 3: The inclusions $x(1,i-1) \subset x(1,j-1)$ and $S(1,j-1) \subset S(1,i-1)$ hold for all $i, j \in T$, $1 < i < j \leq n$.

Proof: The optimal solutions fulfill obviously $x(1,i-1) \subset x(1,j-1)$ if $i > 1$. If the second solution is optimal for $s' \in S(1,j-1)$ then this is also true for the first solution, i. e. $s' \in S(1,i-1)$. □

For illustration example 2 can be used:
The optimal solution is $x = (3,3,0,4,0,3)$ with $T = \{1,2,4,6\}$. Then $x(1,1) = (3) \subset x(1,3) = (3,3,0) \subset x(1,5) = (3,3,0,4,0)$ and $S(1,1) = [0,+\infty) \supset S(1,3) = [1,3] \supset S(1,5) = [2,3]$, but $S(1,2) = [2,+\infty)$.

Under the same conditions as in lemma 2 the existence of monotonous triples can be proved.

Theorem 2: (i) Let $M(1,n)$ be considered and $r-1$ be a planning horizon for the forecast horizon t and $n \geq t \geq r$. Then $S(1,n) \subset S(1,r-1) \cap S(r,n)$ (10)
holds. (ii) Let additionally $r = t$ or let the period $t-1$ be an ordinary planning horizon for $M(1,n)$. Then $S(1,t) \cap S(r,n) \subset S(1,n)$ (11)
holds. (iii) Let the conditions from (i) and (ii) be fulfilled and
let furthermore $S(1,t) \supset S(1,r-1)$ (12)

be true. Then $S(1,n) = S(1,r\text{-}1) \cap S(r,n)$ (13)
holds.

Proof: The relation (10) is a consequence of lemma 2.
Let the inclusion (11) be studied: The case $r = t$ has been proved in [10]. Let $t > r$: The period $t\text{-}1$ itself is an ordinary planning horizon. For this case the relation
$S(1,t) \cap S(t,n) \subset S(1,n)$ has been proved in [11] as well. Let the problem $M(r,n)$ be considered. Since r is setup period for the problem $M(1,n)$ the property of the period $t\text{-}1$ to be ordinary planning horizon does not depend on the periods before r. Hence the period $t\text{-}1$ is also ordinary planning horizon for $M(r,n)$. Then lemma 2 can be applied and $S(r,n) \subset S(t,n)$ holds, i. e. $S(1,t) \cap S(r,n) \subset S(1,t) \cap S(t,n) \subset S(1,n)$ is true.
The relations (10) - (12) yield immediately the identity (13). ☐

Remarks: (i) If $r\text{-}1$ is an ordinary planning horizon, then theorem 2 coincides with the results presented in [11].
(ii) It can be seen that monotonous pairs and triples appear together and one of the notions seems to be redundant under the conditions of the theorem. However, the monotonicity of pairs of stability regions shows the model to become less robust with increasing time horizon, while (strictly) monotonous triples represent a more regular type of monotonicity.
(iii) If $t\text{-}1$ is not an ordinary planning horizon the relation (11) might not hold. Let $n = 6$ in the example 5. Then $r\text{-}1=3$ is a planning horizon for the forecast horizon $t=5$. The period $t\text{-}1=4$ is obviously not an ordinary planning horizon and
$S(1,5) \cap S(4,6) = [6,18] \cap [6,+\infty) \not\subset [9,27] = S(1,6).$
(iv) Unfortunately, the inclusion (12) which guarantees strictly monotonous triples will not hold in general. Sufficient conditions for (12) will be discussed in the next section.
(v) The relations (10) and (11) may all be distinct as the next example will display.

Example 6: Let $d = (7, 1, 18, 6, 1, 2)$, $s = 5$. Then the optimal solutions $x(1,3) = (8,0,18)$, $x(1,4) = (8,0,18,6)$ and $x(1,6) = (8, 0, 18, 9, 0, 0)$ are found and $i(1) = i(2) = 1$, $i(3) = 3$, $i(4) = i(5) = i(6) = 4$. Then period $r\text{-}1 = 3$ is an ordinary planning horizon and the case $r = t$ of the theorem occurs. However, the stability regions $S(4,6) = [4,+\infty)$, $S(1,3) = [1,36]$, $S(1,4) = [1,6]$ and $S(1,6) = [4,9]$ are quite different and the inclusions (10) and (11) are really not identities: $[4,6] \subset [4,9] \subset [4,36]$.

2.3. Sufficient Conditions for Strictly Monotonous Triples

Now, it will be discussed in which cases the inclusion (12) securing the existence of strictly monotonous triples is valid. Here, only the case of $r = t$, i. e. the case of ordinary planning horizons is regarded.

First, let an ordinary planning horizon t-1 for a model $M(1,t)$ be considered and let J_t denote the set of all setup cost inputs for which the appropriate optimal solutions have the planning horizon t-1. This set can be characterized by the following lemma.

Lemma 4: Let s^+ be the largest upper bound of all the stability regions of those optimal solutions for $M(1,t)$ which satisfy the relation $i(t) = t$. (14)
Then $J_t = [0,s^+]$ holds. (15)

Proof: When moving from one stability region to the next and increasing s it can be found that the number of setups $k(s)$ of the corresponding optimal solution will not increase (compare [4]), i.e. $k(s) \geq k(s')$, holds if $s < s'$.
The last setup period $i(t)$ from recursion (5) is also depending on s and k and it will not increase with growing s or falling k (compare [2]). As a consequence, if the notation $i(t,s)$ is used, the relation $i(t,s) \geq i(t,s')$, if $0 \leq s < s'$ (16)
holds. Then (16) immediately yields the formula (15). □

This set is $J_3 = [0,1]$ in the example 1 (comp. Tab. 1). Let $M(1,3)$ in example 6 be considered. The solutions $(7,1,18)$ and $(8,0,18)$ guarantee $i(3) = 3$. The stability regions for both the solutions are $[0, 1]$ and $[1,36]$, respectively. Then $J_3 = [0,36] = [0, 1] \cup [1,36]$.

Lemma 5: The following statements are equivalent:

$$S(1,t\text{-}1) = S(1,t) \wedge S(1,t) \subset J_t \qquad (17)$$

$$S(1,t\text{-}1) \subset J_t. \qquad (18)$$

Proof: (18) $\Rightarrow$ (17): Let $x(1,t)$ be the optimal solution for $M(1,t)$ with the initial cost parameter $s \in S(1,t\text{-}1)$. The period t-1 is an ordinary planning horizon.
According to lemma 2 the inclusion $S(1,t) \subset S(1,t\text{-}1)$ holds.
Let now $s' \in S(1,t\text{-}1) \subset J_t$. It has to be shown that $s' \in S(1,t)$ holds. It is clear that t-1 is a planning horizon for s', too. Therefore an optimal solution for $M(1,t)$ with the parameter s' will have a setup in the period t. Then an optimal solution $x(1,t)$ for $M(1,t)$ with s' is determined by the optimal solution $x(1,t\text{-}1)$ and by adding one setup to $x(1,t\text{-}1)$. Hence, $s' \in S(1,t)$.
The other direction of the proof is quite obvious. ☐

Hence, the condition $S(1,t) \supset S(1,t\text{-}1)$ securing the relation (13) for in theorem 2 is fulfilled if and only if the cost inputs from $S(1,t\text{-}1)$ provide an optimal solution of $M(1,t)$ with $i(t) = t$. In other words, monotonous triples occur if the stability set J_t of the planning horizon t is sufficiently large and contains the set $S(1,t\text{-}1)$.

Let the example 6 be analysed once more: The set $S(1,3)= [1,36]$ is not a subset of $J_4 = [0,6]$. Thus the property (18) cannot be guaranteed.

Unfortunately, the condition (18) (or (12)) is not necessary for relation (13).

Example 7: Let $s = 5$ and $d = (4, 2, 5, 3, 4)$. Then $S(1,2) = [2,+\infty)$, $S(1,3) = [2,10]$, $S(1,5)= = S(3,5) = [3,8]$ and $r=t=3$ and $j=5$. For this example the relation (13) holds,
i. e. $S(1,3) \cap S(3,5) = S(1,5)$, although the properties (12) and (18) are not satisfied:
$S(1,2) \not\subset S(1,3)$ and $S(1,2) \not\subset J_3 = [0,12]$.

2.4. The Size of the Stability Regions and the Time Horizon

Let the example 5, Tab. 3 and Fig. 2 be considered once more. The periods $r-1 = 3m$ are planning horizons for the forecast horizons $t = 3m+2$, $m=1,2,...$. The average (per-period) minimal cost occuring with the use of the time horizon $r-1$ is $(10 + 3 + 6)m/3m = 6.33...$. This value is minimal for the given problem. If the time horizon to be used in the lot size problem can be $r-1=3$, then this horizon is accompanied by the stability region $S(1,3) = [6,+\infty)$, which is probably large enough. If, however, planning procedures do not allow such a small time horizon, the next optimal time horizon is $r-1 = 6$ with the stability region $S(1,6) = [9,27]$. Then, perhaps the time horizon 5 will be prefered with higher average minimal cost 6.4 but with a better stability region $S(1,5) = [6,18]$. This region might be prefered since it leaves more space for equiliteral setup cost variations.

Conclusions

With growing time horizon stability regions of the dynamic lot size problem will behave quite different. Monotonous pairs and triples of stability regions and monotonous pairs of optimal solutions can be identified if planning and forecast horizons exist. Strictly monotonous triples appear if the ordinary planning horizon has a sufficiently large stability set. Concerning the practical application of the dynamic lot size model the analysis makes clear that the robustness of optimal solutions depends on the demand pattern. In many cases a larger time horizon will reduce this robustness and sometimes this robustness is as small as in the worst subintervals of the time horizon. However, nonmonotonous behavior of the stability regions can also be observed.
The size of the stability region can serve as one of the criteria for the determination of the time horizon of the dynamic lot size problem. It can happen that an economically preferable time horizon generates a rather unstable solution. Then it is perhaps worth considering other time horizons with a more preferable stability region.

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Tables

Stability region	*[0,1]*	*[1,3]*	*[3,+∞)*
optimal solution	*(3,2,1)*	*(3,3,0)*	*(6,0,0)*
k(s)	*3*	*2*	*1*
H(s)	*0*	*1*	*4*
i(3)	*3*	*2*	*1*

Tab. 1: Stability regions and other relevant information

t	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*S(1,t)*
x(1,2)	*26*	*0*							*[9,+∞)*
x(1,3)	*26*	*0*	*12*						*[9,24]*
x(1,4)	*26*	*0*	*22*	*0*					*[10,44]*
x(1,5)	*26*	*0*	*31*	*0*	*0*				*[19,62]*
x(1,6)	*26*	*0*	*22*	*0*	*16*	*0*			*[10,30]*
x(1,7)	*26*	*0*	*22*	*0*	*25*	*0*	*0*		*[18,39]*
x(1,8)	*26*	*0*	*22*	*0*	*16*	*0*	*14*	*0*	*[10,25]*

Tab. 2: Optimal solutions associated with the growing time horizon

t	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	f_t/t	*S(1,t)*
x(1,2)	*6*	*0*								*13/2=6.5*	*[3,∞)*
x(1,3)	*9*	*0*	*0*							*19/3=6.33*	*[6,∞)*
x(1,4)	*6*	*0*	*6*	*0*						*26/4=6.4*	*[3,12]*
x(1,5)	*9*	*0*	*0*	*6*	*0*					*32/5=6.4*	*[6,18]*
x(1,6)	*9*	*0*	*0*	*9*	*0*	*0*				*38/6=6.33*	*[9,27]*
x(1,7)	*9*	*0*	*0*	*6*	*0*	*6*	*0*			*45/7=6.43*	*[6,12]*
x(1,8)	*9*	*0*	*0*	*9*	*0*	*0*	*6*	*0*		*51/8=6.38*	*[9,25]*
x(1,9)	*9*	*0*	*0*	*9*	*0*	*0*	*9*	*0*	*0*	*57/9=6.33*	*[9,21]*

Tab. 3: Optimal solutions, minimal costs and stability regions for example 5

Figures

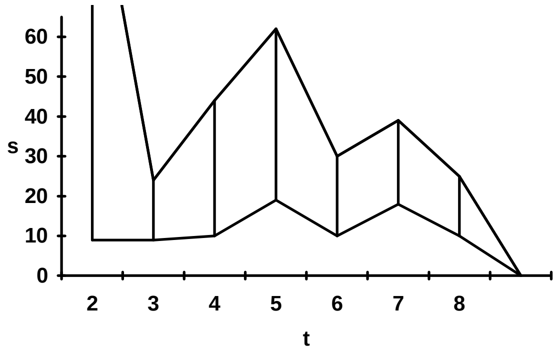

Fig. 1: Stability regions *S(1,t)* for example 4

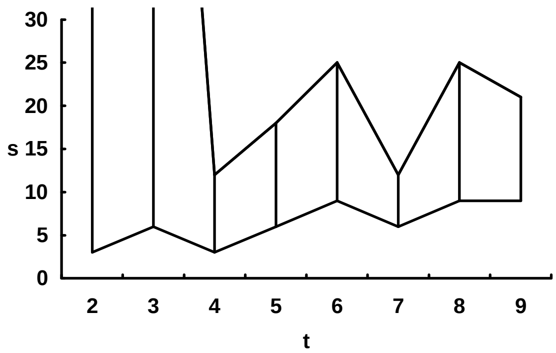

Fig. 2: Stability regions *S(1,t)* and average minimal cost f_t/t for example 5